JOURNAL OF APPROXIMATION THEORY 76, 65–76 (1994)

Average Widths of Sobolev Classes on $\mathbb{R}^n$

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Communicated by Allan Pinkus

Received February 21, 1992; revised July 23, 1992

In this paper, we introduce the concept of φ-average dimension for some subspaces of $L_{\mathbb{p}}(\mathbb{R}^n)$ and define the corresponding Kolmogorov φ-average ν-width of a set in $L_{\mathbb{p}}(\mathbb{R}^n)$. For the Sobolev class $W_{\mathbb{p}}^{r}(\mathbb{R}^n)$ in $L_{\mathbb{q}}(\mathbb{R}^n)$ we find necessary and sufficient conditions for this quantity to be finite and determine its asymptotic behaviour as $\nu \to \infty$. We also obtain the exact value of the average ν-widths of some classes of functions in $L_2(\mathbb{R}^n)$.

1. Definitions and Formulation of the Main Results

1.1. Let $(X, \|\cdot\|)$ be a normed linear space. We use the following notation:

$BX := \{x \in X \mid \|x\| \leqslant 1\}$ is the unit ball in X,

$\mathrm{Lin}(X)$ is the set of all linear subspaces of X.

$d(x, A, X) := \inf\{\|x-y\| \mid y \in A\}$ is the distance of $x \in X$ from $A \subset X$,

$d(C, A, X) := \sup\{d(x, A, X) \mid x \in C\}$ is the deviation of $C \subset X$ from $A \subset X$,

$d_n(C, X) := \inf\{d(C, L, X) \mid L \in \mathrm{Lin}(X), \dim L \leqslant n\}$ is the Kolmogorov n-width of C in X ($n \in \mathbb{Z}_+ := \{0, 1, 2, ...\}$).

1.2. Let $n \in \mathbb{N} := \{1, 2, ...\}$, $\mathbb{p} := (p_1, ..., p_n)$, $1 \leqslant p_i \leqslant \infty$, $i = 1, ..., n$, $I = (a, b)$, $-\infty \leqslant a < b \leqslant \infty$, $I^n = I \times \cdots \times I$, and $L_{\mathbb{p}}(I^n)$ denote the Banach space of measurable functions $x(\cdot)$ on I^n with the mixed norm

$$\|x(\cdot)\|_{L_{\mathbb{p}}(I^n)} := \left(\int_I dt_n \left(\int_I dt_{n-1} \cdots \left(\int_I |x(t)|^{p_1} dt_1\right)^{p_2/p_1} \cdots\right)^{p_n/p_{n-1}}\right)^{1/p_n}$$

(see [1]).

When $\mathbb{p} = (p, ..., p)$, $L_{\mathbb{p}}(I^n)$ coincides with the usual space $L_p(I^n)$.

For ease of writing, we denote $\mathbb{1} = (1, ..., 1)$, $\mathbb{2} = (2, ..., 2)$, $\boldsymbol{\infty} = (\infty, ..., \infty)$.

* Supported by Russian Foundation of Fundamental Research.

0021-9045/94 $6.00

If $\mathbb{p}=(p_1, ..., p_n)$, $\mathbb{q}=(q_1, ..., q_n)$, $1 \leqslant p_i$, $q_i \leqslant \infty$, $i=1, ..., n$, then $\mathbb{p} \leqslant \mathbb{q}$ ($\mathbb{p} < \mathbb{q}$) means $p_i \leqslant q_i$ ($p_i < q_i$), $i=1, ..., n$.

Let $a>0$, and P_a be the continuous linear operator in $L_{\mathbb{p}}(\mathbb{R}^n)$ defined by $P_a x(\cdot) := X_a(\cdot)\, x(\cdot)$, where $X_a(\cdot)$ is the characteristic function of the cube $[-a, a]^n$.

Set

$$\operatorname{Lin}_c(L_{\mathbb{p}}(\mathbb{R}^n)) := \{L \in \operatorname{Lin}(L_{\mathbb{p}}(\mathbb{R}^n)) \mid \text{restriction } P_a \text{ to } L \text{ is a compact operator for all } a>0\}.$$

Let $L \in \operatorname{Lin}_c(L_{\mathbb{p}}(\mathbb{R}^n))$ and $a>0$. Then $P_a(L \cap BL_{\mathbb{p}}(\mathbb{R}^n))$ is relatively compact and therefore the quantity

$$K_\varepsilon(a, L, L_{\mathbb{p}}(\mathbb{R}^n)) := \min\{n \in \mathbb{Z}_+ \mid d_n(P_a(L \cap BL_{\mathbb{p}}(\mathbb{R}^n)), L_{\mathbb{p}}(\mathbb{R}^n)) < \varepsilon\}$$

is finite for every $a>0$ and $\varepsilon>0$.

It is easily verified that the function $a \to K_\varepsilon(a, L, L_{\mathbb{p}}(\mathbb{R}^n))$ is nondecreasing and the function $\varepsilon \to K_\varepsilon(a, L, L_{\mathbb{p}}(\mathbb{R}^n))$ is nonincreasing.

Remark. Obviously we can identify $P_a(L \cap BL_{\mathbb{p}}(\mathbb{R}^n))$ with the restriction $L \cap BL_{\mathbb{p}}(\mathbb{R}^n)$ to $[-a, a]^n$. It is then easy to check that

$$K_\varepsilon(a, L, L_{\mathbb{p}}(\mathbb{R}^n)) = \min\{n \in \mathbb{Z}_+ \mid d_n(P_a(L \cap BL_{\mathbb{p}}(\mathbb{R}^n)), L_{\mathbb{p}}([-a, a]^n)) < \varepsilon\}.$$

Let Φ denote the set of all positive nondecreasing functions $\varphi(\cdot)$ on $(0, \infty)$ for which $\varphi(a) \to \infty$ as $a \to \infty$.

DEFINITION 1.1. Let $L \in \operatorname{Lin}_c(L_{\mathbb{p}}(\mathbb{R}^n))$ and $\varphi(\cdot) \in \Phi$. Then the φ-average dimension of L in $L_{\mathbb{p}}(\mathbb{R}^n)$ is defined as

$$\dim(L, L_{\mathbb{p}}(\mathbb{R}^n), \varphi(\cdot)) := \lim_{\varepsilon \to 0} \liminf_{a \to \infty} \frac{K_\varepsilon(a, L, L_{\mathbb{p}}(\mathbb{R}^n))}{\varphi(a)}. \tag{1.1}$$

If $\varphi(a)=(2a)^n$ (the volume of cube $[-a, a]^n$) then we call (1.1) the average dimension of L in $L_{\mathbb{p}}(\mathbb{R}^n)$ and denote it by $\overline{\dim}(L, L_{\mathbb{p}}(\mathbb{R}^n))$. In this case (1.1) is a slight modification of the definition given by Tikhomirov [10].

DEFINITION 1.2. Let C be a centrally symmetric subset of $L_{\mathbb{p}}(\mathbb{R}^n)$, $\varphi(\cdot) \in \Phi$, and $\nu \geqslant 0$. The Kolmogorov φ-average ν-width of C in $L_{\mathbb{p}}(\mathbb{R}^n)$ is defined as

$$d_\nu(C, L_{\mathbb{p}}(\mathbb{R}^n), \varphi(\cdot)) := \inf_L \sup_{x(\cdot) \in C} \inf_{y(\cdot) \in L} \|x(\cdot) - y(\cdot)\|_{L_{\mathbb{p}}(\mathbb{R}^n)}, \tag{1.2}$$

where the infimum is taken over all subspaces $L \in \mathrm{Lin}_c(L_{\mathbb{p}}(\mathbb{R}^n))$ such that $\dim(L, L_{\mathbb{p}}(\mathbb{R}^n), \varphi(\cdot)) \leqslant \nu$.

If $\varphi(a) = (2a)^n$, then we call (1.2) the Kolmogorov average ν-width of C in $L_{\mathbb{p}}(\mathbb{R}^n)$ and denote it by $\bar{d}_\nu(C, L_{\mathbb{p}}(\mathbb{R}^n))$.

1.3. Let $S = S(\mathbb{R}^n)$ be the space of rapidly decreasing functions on $\mathbb{R}^n$, and $S' = S'(\mathbb{R}^n)$ the dual space of tempered distributions with the usual topologies. Denote by $F: S' \to S'$ and $F^{-1}: S' \to S'$ the Fourier transform and its inverse, respectively.

For each $\alpha \in \mathbb{R}$, $\mathscr{K}_\alpha: S' \to S'$ denotes the operator defined by $\mathscr{K}_\alpha x := (1 + |\sigma|^2)^{\alpha/2} x$, where $\sigma = (\sigma_1, ..., \sigma_n)$, $|\sigma|^2 = \sigma_1^2 + \cdots + \sigma_n^2$. Let $I_\alpha := F^{-1} \circ \mathscr{K}_\alpha \circ F$ and $\mathbb{1} \leqslant \mathbb{p} \leqslant \boldsymbol{\infty}$. Set

$$\mathscr{H}_{\mathbb{p}}^\alpha(\mathbb{R}^n) := \{x \in S'(\mathbb{R}^n) \mid (I_\alpha x)(\cdot) \in L_{\mathbb{p}}(\mathbb{R}^n)\}.$$

This is a Banach space with norm $\|x(\cdot)\|_{\mathscr{H}_{\mathbb{p}}^\alpha(\mathbb{R}^n)} := \|(I_\alpha x)(\cdot)\|_{L_{\mathbb{p}}(\mathbb{R}^n)}$ [3].

If $\mathbb{p} = (p, ..., p)$, then $\mathscr{H}_p^\alpha(\mathbb{R}^n)$ are the well-known spaces of Bessel potentials, or Liouville spaces (see, for example, [8, 9]).

When $\mathbb{1} < \mathbb{p} < \boldsymbol{\infty}$ and $\alpha = r \in \mathbb{N}$, $\mathscr{H}_{\mathbb{p}}^\alpha(\mathbb{R}^n)$ coincides with the Sobolev space

$$\mathscr{W}_{\mathbb{p}}^\alpha(\mathbb{R}^n) := \{x(\cdot) \in L_{\mathbb{p}}(\mathbb{R}^n) \mid \partial^r x(\cdot)/\partial t_j^r \in L_{\mathbb{p}}(\mathbb{R}^n), j = 1, ..., n\}$$

(see [3]).

The set

$$W_{\mathbb{p}}^r(\mathbb{R}^n) := \left\{x(\cdot) \in \mathscr{W}_{\mathbb{p}}^\alpha(\mathbb{R}^n) \;\middle|\; \sum_{j=1}^n \|\partial^r x(\cdot)/\partial t_j^r\|_{L_{\mathbb{p}}(\mathbb{R}^n)} \leqslant 1\right\}$$

we call the Sobolev class.

In the case where $n = 1$, we have determined the asymptotic behaviour of $d_\nu(W_p^r(\mathbb{R}), L_q(\mathbb{R}), \varphi(\cdot))$ (for some p, q) as ν grows, and we have also found the exact values of $\bar{d}_\nu(W_p^r(\mathbb{R}), L_p(\mathbb{R}))$ for all $1 \leqslant p \leqslant \infty$ (see [5–7]). In this paper, we are interested in the case where n is an arbitrary positive integer.

1.4. The following assertions are the main results in this paper.

THEOREM 1.1. *Let* r, $n \in \mathbb{N}$, $\mathbb{1} < \mathbb{p} = (p_1, ..., p_n) = \mathbb{q} = (q_1, ..., q_n) < \boldsymbol{\infty}$ *or* $\mathbb{1} < \mathbb{p} \leqslant \mathbb{q} \leqslant \mathbb{2}$, *if* $\mathbb{p} \neq \mathbb{q}$. *Next, let* $r > \sum_{j=1}^n (1/p_j - 1/q_j)$, $\varphi(\cdot) \in \Phi$, *and* $\nu > 0$. *Then* $d_\nu(W_{\mathbb{p}}^r(\mathbb{R}^n), L_{\mathbb{q}}(\mathbb{R}^n), \varphi(\cdot)) < \infty$ *if and only if* $\liminf_{a \to \infty}(a^n/\varphi(a)) < \infty$.

If, in addition, $\liminf_{a \to \infty}(a^n/\varphi(a)) > 0$, then

$$d_\nu(W_{\mathbb{p}}^r(\mathbb{R}^n), L_{\mathbb{q}}(\mathbb{R}^n), \varphi(\cdot)) \asymp \begin{cases} \nu^{-r/n}, & \mathbb{p} = \mathbb{q} \\ \nu^{-(1/n)(r - \sum_{j=1}^n (1/p_j - 1/q_j))}, & \mathbb{1} < \mathbb{p} \leqslant \mathbb{q} \leqslant \mathbb{2}. \end{cases}$$

THEOREM 1.2. *Let $n \in \mathbb{N}$, $\alpha > 0$, and $\nu \geqslant 0$. Then*

$$\bar{d}_\nu(B\mathscr{H}_2^\alpha(\mathbb{R}^n), L_2(\mathbb{R}^n)) = \left(1 + 4\pi \left(\Gamma\left(\frac{n}{2}+1\right)\nu\right)^{2/n}\right)^{-\alpha/2},$$

where $\Gamma(\cdot)$ is the Euler function (and, recall, that $B\mathscr{H}_2^\alpha(\mathbb{R}^n)$ is the unit ball in $\mathscr{H}_2^\alpha(\mathbb{R}^n)$).

2. PRELIMINARY RESULTS

Let $n \in \mathbb{N}$, $\sigma = (\sigma_1, \dots \sigma_n) \geqslant \mathbb{0}$, $\mathbb{1} \leqslant \mathbb{p} \leqslant \boldsymbol{\infty}$, and $\mathscr{B}_{\sigma, \mathbb{p}}(\mathbb{R}^n)$ denote the restriction to $\mathbb{R}^n$ of the space of all functions of exponential type σ which belong to $L_\mathbb{p}(\mathbb{R}^n)$ (see [8]).

LEMMA 2.1. *Let $\mathbb{1} \leqslant \mathbb{p} \leqslant \boldsymbol{\infty}$ and $\sigma = (\sigma_1, \dots, \sigma_n) > \mathbb{0}$. Then $\mathscr{B}_{\sigma, \mathbb{p}}(\mathbb{R}^n) \in \mathrm{Lin}_c(\mathbb{R}^n)$ and*

$$\overline{\dim}(\mathscr{B}_{\sigma, \mathbb{p}}(\mathbb{R}^n), L_\mathbb{p}(\mathbb{R}^n)) \leqslant \frac{\sigma_1 \cdot \cdots \cdot \sigma_n}{\pi^n}.$$

The case for $\mathbb{p} = (p, \dots, p)$ follows from [2]. The argument in the general case is similar.

LEMMA 2.2. *Let $\mathbb{1} < \mathbb{p} = (p_2, \dots, p_n) \leqslant \mathbb{q} = (q_1, \dots, q_n) < \boldsymbol{\infty}$, $r \in \mathbb{N}$, $r > \sum_{i=1}^n (1/p_i - 1/q_i)$, $\gamma > 0$, and $\sigma = (\gamma^{1/n}, \dots, \gamma^{1/n})$. Then there exists a constant $c > 0$ depending only $\mathbb{p}$, $\mathbb{q}$, and r so that*

$$d(W_\mathbb{p}^r(\mathbb{R}^n), \mathscr{B}_{\sigma, \mathbb{q}}(\mathbb{R}^n), L_\mathbb{q}(\mathbb{R}^n)) \leqslant c\gamma^{-(1/n)(r - \sum_{j=1}^n (1/p_j - 1/q_j))}.$$

This is a consequence of the general result [4].

Let $\mathscr{I}$ be a finite set, $n \in \mathbb{N}$, $\mathscr{I}^n = \mathscr{I} \times \cdots \times \mathscr{I}$, $N = \operatorname{card} \mathscr{I}^n$, and $\mathbb{1} \leqslant \mathbb{p} = (p_1, \dots, p_n) \leqslant \boldsymbol{\infty}$. Denote by $l_\mathbb{p}^N(\mathscr{I}^n)$ the normed linear space of functions $a_{j_1 \cdots j_n}$, $j_k \in \mathscr{I}$, $1 \leqslant k \leqslant n$, on $\mathscr{I}^n$ with the mixed norm

$$\|a_{j_1, \dots, j_n}\|_{l_\mathbb{p}^N(\mathscr{I}^n)} := \left(\sum_{j_n}\left(\sum_{j_{n-1}} \cdots \left(\sum_{j_1} |a_{j_1, \dots, j_n}|^{p_1}\right)^{p_2/p_1} \cdots\right)^{p_n/p_{n-1}}\right)^{1/p_n}.$$

LEMMA 2.3. *Let k, $n \in \mathbb{N}$, $\mathscr{I}$ be a finite set, $\operatorname{card} \mathscr{I}^n =: N > k$, $\mathbb{1} \leqslant \mathbb{p} \leqslant \mathbb{q} \leqslant 2$, and $Bl_\mathbb{p}^N(\mathscr{I}^n)$ the unit ball in $l_\mathbb{p}^N(\mathscr{I}^n)$. Then*

$$d_k(Bl_\mathbb{p}^N(\mathscr{I}^n), l_\mathbb{q}^N(\mathscr{I}^n)) \geqslant \sqrt{1 - \frac{k}{N}}.$$

Proof. The inequality

$$\|a_{j_1 \cdots j_n}\|_{l_{\mathfrak{q}}^N(\mathscr{I}^n)} \leqslant \|a_{j_1 \cdots j_n}\|_{l_{\mathfrak{p}}^N(\mathscr{I}^n)} \tag{2.1}$$

holds true for all $a_{j, \ldots, j_n} \in l_{\mathfrak{p}}^N(\mathscr{I}^n)$.

Indeed, if $n = 1$, then the assertion is true (see [12]). The general case is proved by an obvious inductive argument.

By (2.1) and definition of the Kolmogorov k-width and since $\mathbb{1} \leqslant \mathfrak{p} \leqslant \mathfrak{q} \leqslant \mathbb{2}$, we obtain

$$d_k(Bl_{\mathfrak{p}}^N(\mathscr{I}^n), l_{\mathfrak{q}}^N(\mathscr{I}^n)) \geqslant d_k(Bl_{\mathbb{1}}^N(\mathscr{I}^n), l_{\mathbb{2}}^N(\mathscr{I}^n)).$$

$l_{\mathfrak{p}}^N(\mathscr{I}^n)$ in the special case $\mathfrak{p} = (p, \ldots, p)$ we can identify with $l_p(\mathbb{R}^N)$. It is the normed space of vectors $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$ with norm $\|\xi\|_{l_p(\mathbb{R}^N)} := (\sum_{i=1}^n |\xi_i|^p)^{1/p}$. Thus,

$$d_k(Bl_{\mathbb{1}}^N(\mathscr{I}^n), l_{\mathbb{2}}^N(\mathscr{I}^n)) = d_k(Bl_1(\mathbb{R}^N), l_2(\mathbb{R}^N)) = \sqrt{1 - \frac{k}{N}},$$

where the last equality is a well-known result (see, for example, [11]). Lemma 2.3 is proved.

Let $\psi(\cdot) \in C^\infty(\mathbb{R})$, supp $\psi(\cdot) \subset [0, 1]$, $\psi(t) \geqslant 0$, $\int_0^1 \psi(t)\, dt = 1$, and $h > 0$. Put $\psi_{j,h}(t) = \psi(t/h - j)$, $j \in \mathbb{Z}$. Then supp $\psi_{j,h}(\cdot) \subset \Delta_{j,h} := [jh, (j+1)h]$.

Let $n \in \mathbb{N}$. We associate with any $(j_1, \ldots, j_n) \in \mathbb{Z}^n$ and $h > 0$ the following function on $\mathbb{R}^n$:

$$\Psi_{j_1 \cdots j_n, h}(t_1, \ldots, t_n) := \prod_{k=1}^n \psi_{j_k, h}(t_k).$$

It is obvious that $\Psi_{j_1 \cdots j_n, h}(\cdot) \in C^\infty(\mathbb{R}^n)$ and supp $\Psi_{j_1 \cdots j_n, h}(\cdot) \subset \Delta_{j_1 \cdots j_n, h} := \Delta_{j_1, h} \times \cdots \times \Delta_{j_n, h}$.

For any $n, m \in \mathbb{N}$, $h > 0$ define the space $L_{m,h}(n)$ by

$$L_{m,h}(n) := \operatorname{span}\{\Psi_{j_1 \cdots j_n, h}(\cdot) \mid j_1, \ldots, j_n = -m, \ldots, m-1\}.$$

It is easy to see that $\dim L_{m,h}(n) = (2m)^n$, and supp $x(\cdot) \subset [-mh, mh]^n$ when $x(\cdot) \in L_{m,h}(n)$.

For $x(\cdot) \in L_1([-mh, mh]^n)$ put

$$P_{m,n,h} x(\cdot) := h^{-n} \sum_{j_1, \ldots, j_n = -m}^{m-1} \left(\int_{\Delta j_1 \cdots j_n, h} x(\tau)\, d\tau \right) \Psi_{j_1 \cdots j_n, h}(\cdot). \tag{2.2}$$

LEMMA 2.4. *Let $m, n \in \mathbb{N}$ and $h > 0$.*

(1) *If $\mathbb{1} \leqslant \mathfrak{p} = (p_1, \ldots, p_n) \leqslant \boldsymbol{\infty}$, then $P_{m,n,h}$ is a continuous linear projection in $L_{\mathfrak{p}}([-mh, mh]^n)$, and there exists a constant $c > 0$ depending only on $\mathfrak{p}$ such that $\|P_{m,n,h}\| \leqslant c$.*

(2) *If* $\mathbb{1} \leqslant \mathbb{p} = (p_1, ..., p_n) \leqslant \mathbb{\infty}$, $k = (k_1, ..., k_n) \in \mathbb{Z}^n_+$, $\mathcal{I}_m = \{-m, ..., m-1\}$, $N = (2m)^n$, *and* $x(\cdot) = \sum_{j_1, ..., j_n = -m}^{m-1} a_{j_1 \cdots j_n} \Psi_{j_1 \cdots j_n, h}(\cdot) \in L_{m, h}(n)$, *then there exists a constant* $c > 0$ *depending only on* $\mathbb{p}$ *and* k *such that*

$$\|\partial^{k_1 + \cdots + k_n} x(\cdot)/\partial t_1^{k_1} \cdots \partial t_n^{k_n}\|_{L_{\mathbb{p}}([-mh, mh]^n)} = ch^{-|k| + \sum_{j=1}^n (1/p_j)} \|a_{j_1 \cdots j_n}\|_{l^N_{\mathbb{p}}(\mathcal{I}^n_m)}, \tag{2.3}$$

where $|k| = k_1 + \cdots + k_n$.

(3) *If* $\mathbb{1} \leqslant \mathbb{p} = (p_1, ..., p_n) \leqslant \mathbb{q} = (q_1, ..., q_n) \leqslant \mathbb{\infty}$ *and* $k = (k_1, ..., k_n) \in \mathbb{Z}^n_+$, *then there exists a constant* $c > 0$ *depending only on* $\mathbb{p}$, $\mathbb{q}$, *and* k *such that the inequality*

$$\|\partial^{k_1 + \cdots + k_n} x(\cdot)/\partial t_1^{k_1} \cdots \partial t_n^{k_n}\|_{L_{\mathbb{q}}([-mh, mh]^n)} \leqslant ch^{-|k| + \sum_{j=1}^n (1/q_j - 1/p_j)} \|x(\cdot)\|_{L_{\mathbb{p}}([-mh, mh]^n)} \tag{2.4}$$

holds true for all $x(\cdot) \in L_{m, h}(n)$.

The assertions of Lemma 2.4 are directly verified for $n = 1$. The general case is proved by an inductive argument. We omit the corresponding routine calculations.

3. Proofs of the Main Results

3.1. *Proof of Theorem* 1.1. *Necessity.* Let $d_v(W^r_{\mathbb{p}}(\mathbb{R}^n), L_{\mathbb{q}}(\mathbb{R}^n), \varphi(\cdot)) < \infty$. Then $d(W^r_{\mathbb{p}}(\mathbb{R}^n), L, L_{\mathbb{q}}(\mathbb{R}^n)) < \infty$ for some $L \in \mathrm{Lin}_c(L_{\mathbb{q}}(\mathbb{R}^n))$ such that $\dim(L, L_{\mathbb{q}}(\mathbb{R}^n), \varphi(\cdot)) \leqslant v$.

Let $\varepsilon > 0$. There is a sequence $\{a_s\}_{s \in \mathbb{N}}$ for which

$$\liminf_{a \to \infty} \frac{K_\varepsilon(a, L, L_{\mathbb{q}}(\mathbb{R}^n))}{\varphi(a)} = \lim_{s \to \infty} \frac{K_\varepsilon(a_s, L, L_{\mathbb{q}}(\mathbb{R}^n))}{\varphi(a_s)}.$$

For each $s \in \mathbb{N}$ there exists an $M(s, \varepsilon) \in \mathrm{Lin}(L_{\mathbb{q}}([-a_s, a_s]^n)$ so that $\dim M(s, \varepsilon) \leqslant K_\varepsilon(a_s, L, L_{\mathbb{q}}(\mathbb{R}^n))$ and

$$d(P_{a_s} y(\cdot), M(s, \varepsilon), L_{\mathbb{q}}([-a_s, a_s]^n)) < \varepsilon \, \|y(\cdot)\|_{L_{\mathbb{q}}(\mathbb{R}^n)} \tag{3.1}$$

for all $y(\cdot) \in L$.

Set $m_s := [(4v\varphi(a_s))^{1/n}/2]$, $h_s := 2a_s/(4v\varphi(a_s))^{1/n}$ and denote $L_s := L_{m_s, h_s}(n)$ (see Section 2). Since $m_s h_s \leqslant \alpha_s$, then $\sup x(\cdot) \subset [-a_s, a_s]^n$ for $x(\cdot) \in L_s$.

Let $c(k, \mathbb{p})$ be a constant in (2.4) for $\mathbb{p} = \mathbb{q}$, $c_1 := c_1(r, \mathbb{p}) := n \max\{c(k, \mathbb{p}) \mid |k| = r\}$, $s \in \mathbb{N}$, and

$$x(\cdot) \in C_1^{-1} h_s^r L_s \cap BL_{\mathbb{p}}([-a_s, a_s]^n). \tag{3.2}$$

Then, by (2.4), $x(\cdot)\in W^r_{\mathbf{p}}(\mathbb{R}^n)$ (we assume that $x(t)=0$ for t outside the $[-a_s, a_s]^n$).

For each $y(\cdot)\in L$, we have ($M := M(s, \varepsilon)$)

$$\begin{aligned}\|x(\cdot)-y(\cdot)\|_{L_{\mathbf{q}}(\mathbb{R}^n)} &\geqslant \|x(\cdot)-P_{a_s}y(\cdot)\|_{L_{\mathbf{q}}([-a_s,a_s]^n)}\\ &\geqslant d(x(\cdot), M, L_{\mathbf{q}}([-a_s, a_s]^n))\\ &\quad - d(P_{a_s}y(\cdot), M, L_{\mathbf{q}}([-a_s, a_s]^n))\\ &\overset{(3.1)}{\geqslant} d(x(\cdot), M, L_{\mathbf{q}}([-a_s, a_s]^n)) - \varepsilon\,\|y(\cdot)\|_{L_{\mathbf{q}}(\mathbb{R}^n)}\\ &\geqslant d(x(\cdot), M, L_{\mathbf{q}}([-a_s, a_s]^n) - \varepsilon\,\|x(\cdot)\|_{L_{\mathbf{q}}(\mathbb{R}^n)}\\ &\quad - \varepsilon\,\|x\cdot) - y(\cdot)\|_{L_{\mathbf{q}}(\mathbb{R}^n)},\end{aligned}$$

i.e.,

$$(1+\varepsilon)\,\|x(\cdot)-y(\cdot)\|_{L_{\mathbf{q}}(\mathbb{R}^n)} \geqslant d(x(\cdot), M, L_{\mathbf{q}}([-a_s, a_s]^n)) - \varepsilon\,\|x(\cdot)\|_{L_{\mathbf{q}}(\mathbb{R}^n)}.$$

It follows that

$$\begin{aligned}(1+\varepsilon)\, &d(W^r_{\mathbf{p}}(\mathbb{R}^n), L, L_{\mathbf{q}}(\mathbb{R}^n))\\ &\geqslant d(x(\cdot), M, L_{\mathbf{q}}([-a_s, a_s]^n)) - \varepsilon\,\|x(\cdot)\|_{L_{\mathbf{q}}(\mathbb{R}^n)}.\end{aligned} \tag{3.3}$$

By (2.4) for $k=0$ and (3.2), one has

$$\begin{aligned}\|x(\cdot)\|_{L_{\mathbf{q}}(\mathbb{R}^n)} &= \|x(\cdot)\|_{L_{\mathbf{q}}([-m_s h_s, m_s h_s]^n)}\\ &\leqslant C_2 h_s^{\sum_{j=1}^n (1/q_j - 1/p_j)}\,\|x(\cdot)\|_{L_{\mathbf{p}}([-a_s, a_s]^n)}\\ &\leqslant c_3 h_s^{\sum_{j=1}^n (1/q_j - 1/p_j)+r},\end{aligned} \tag{3.4}$$

where $c_3>0$ depends only $\mathbf{p}$, $\mathbf{q}$, and r.

By taking the supremum over $x(\cdot)$ satisfying (3.2), and using (3.4), we deduce from (3.3) that

$$\begin{aligned}(1+\varepsilon)\, &d(W^r_{\mathbf{p}}(\mathbb{R}^n), L, L_{\mathbf{q}}(\mathbb{R}^n))\\ &\geqslant c_1^{-1} h_s^r d(L_s \cap BL_{\mathbf{p}}([-a_s, a_s]^n), M, L_{\mathbf{q}}([-a_s, a_s]^n))\\ &\quad - \varepsilon c_3 h_s^{r+\sum_{j=1}^n (1/q_j - 1/p_j)}.\end{aligned} \tag{3.5}$$

For sufficiently large s, $K_\varepsilon(a_s, L, L_{\mathbf{q}}(\mathbb{R}^n)) \leqslant \frac{3}{2}\nu\varphi(a_s)$ and since $\frac{3}{2}\nu\varphi(a_s)/\dim L_s \to \frac{3}{8}$, then there is an $s_0 = s_0(\varepsilon)\in\mathbb{N}$ so that $\frac{3}{2}\nu\varphi(a_s)/\dim L_s \leqslant \frac{1}{2}$, $s\geqslant s_0$. Thus,

$$K_\varepsilon(a_s, L, L_{\mathbf{q}}(\mathbb{R}^n)) \leqslant \tfrac{1}{2}\dim L_s \tag{3.6}$$

for all $s\geqslant s_0$.

Let $s \geqslant s_0$. Put $l_s := 2^{n-1}[(4v\varphi(a_s))^{1/n}/2]$. Then

$$l_s = \tfrac{1}{2} \dim L_s \overset{(3.6)}{\geqslant} K_\varepsilon(a_s, L, L_{\mathfrak{q}}(\mathbb{R}^n)) \geqslant \dim M(s, \varepsilon).$$

It follows from this and the first two assertions of Lemma 2.4 (by using the usual discretization technique) that

$$\begin{aligned} d(L_s \cap BL_{\mathfrak{p}}([-a_s, a_s]^n), M(s, \varepsilon), L_{\mathfrak{q}}([-a_s, a_s]^n)) \\ \geqslant d_{l_s}(L_s \cap BL_{\mathfrak{p}}([-a_s, a_s]^n), L_{\mathfrak{q}}([-a_s, a_s]^n)) \\ \geqslant c_4 d_{l_s}(L_s \cap BL_{\mathfrak{p}}([-a_s, a_s]^n), L_s \cap L_{\mathfrak{q}}([-a_s, a_s]^n)) \\ \geqslant c_5 h_s^{\sum_{i=1}^n (1/q_i - 1/p_i)} d_{l_s}(Bl_{\mathfrak{p}}^{N_s}(\mathcal{I}_{m_s}^n), l_{\mathfrak{q}}^{N_s}(\mathcal{I}_{m_s}^n)), \end{aligned} \tag{3.7}$$

where $\mathcal{I}_{m_s} = \{-m_s, ..., m_s - 1\}$, $N_s = (2m_s)^n$.

From (3.5), (3.7), and Lemma 2.3, we get

$$\begin{aligned} (1+\varepsilon)\, d(W_{\mathfrak{p}}^r(\mathbb{R}^n), L, L_{\mathfrak{q}}(\mathbb{R}^n)) \\ \geqslant (c_6 - \varepsilon c_3)\, h_s^{r + \sum_{j=1}^n (1/q_j - 1/p_j)} \\ = c_7(c_6 - \varepsilon c_3)\, v^{-(1/n)(r - \sum_{j=1}^n (1/p_j - 1/q_j))} (a_s^n/\varphi(a_s))^{(1/n)(r - \sum_{j=1}^n (1/p_j - 1/q_j))}. \end{aligned} \tag{3.8}$$

For sufficiently small $\varepsilon > 0$, $c_6 - \varepsilon c_3 > 0$. Since $r - \sum_{j=1}^n (1/p_j - 1/q_j) > 0$ and the left-hand side of (3.8) is finite, then $\liminf_{a \to \infty} (a^n/\varphi(a)) < \infty$. The necessity is proved.

Sufficiency. Let $\liminf_{a \to \infty} (a^n/\varphi(a)) =: b < \infty$. From the identity

$$\frac{K_\varepsilon(a, \mathcal{B}_{\sigma, \mathfrak{q}}(\mathbb{R}^n), L_{\mathfrak{q}}(\mathbb{R}^n))}{\varphi(a)} = \frac{K_\varepsilon(a, \mathcal{B}_{\sigma, \mathfrak{q}}(\mathbb{R}^n), L_{\mathfrak{q}}(\mathbb{R}^n))}{(2a)^n} \cdot \frac{(2a)^n}{\varphi(a)}$$

and Lemma 2.1 it follows that (for $\sigma = (\gamma^{1/n}, ..., \gamma^{1/n})$)

$$\dim(\mathcal{B}_{\sigma, \mathfrak{q}}(\mathbb{R}^n), L_{\mathfrak{q}}(\mathbb{R}^n), \varphi(\cdot)) \leqslant \gamma 2^n b/\pi^n. \tag{3.9}$$

Put $\gamma = v(\pi/2)^n b^{-1}$ if $b > 0$, and $\gamma = 1$ if $b = 0$. Then, by (3.9) and Lemma 2.2, we have

$$d_v(W_{\mathfrak{p}}^r(\mathbb{R}^n), L_{\mathfrak{q}}(\mathbb{R}^n), \varphi(\cdot)) \leqslant c_1 v^{-(1/n)(r - \sum_{j=1}^n (1/p_j - 1/q_j))},$$

that is, $d_v(W_{\mathfrak{p}}^r(\mathbb{R}^n), L_{\mathfrak{q}}(\mathbb{R}^n), \varphi(\cdot)) < \infty$ and, in addition, the required upper bound is obtained.

Let $\liminf_{a \to \infty} (a^n/\varphi(a)) > 0$. Then the required lower bound follows from (3.8). Theorem 1.1 is proved.

3.2. *Proof of Theorem* 1.2. *The upper bound.* Let $\rho > 0$, $\rho B\mathbb{R}^n := \{t = (t_1, ..., t_n) \in \mathbb{R}^n \mid t_1^2 + \cdots + t_n^2 \leqslant \rho^2\}$ and F be the Fourier transform in $L_2(\mathbb{R}^n)$. Denote by $\mathscr{G}_\rho(\mathbb{R}^n)$ the set of functions in $L_2(\mathbb{R}^n)$ whose Fourier transform is contained in $\rho B\mathbb{R}^n$. Then $\mathscr{G}_\rho(\mathbb{R}^n) \in \mathrm{Lin}_c(L_2(\mathbb{R}^n))$ and

$$\overline{\dim}(\mathscr{G}_\rho(\mathbb{R}^n), L_2(\mathbb{R}^n)) = V_n(\rho)/(2\pi)^n, \tag{3.10}$$

where $V_n(\rho) := \pi^{n/2}\rho^n/\Gamma(n/2+1)$ is the volume of $\rho B\mathbb{R}^n$.

This assertion follows from [2], where the more general formula was proved.

By T_ρ denote the map in $L_2(\mathbb{R}^n)$ defined by $FT_\rho x(\cdot) := X_\rho Fx(\cdot)$, where $X_\rho(\cdot)$ is the characteristic function of $\rho B\mathbb{R}^n$. It is not hard to check that T_ρ is a continuous linear operator in $L_2(\mathbb{R}^n)$.

Let $x(\cdot) \in B\mathscr{H}_2^\alpha(\mathbb{R}^n)$. By Plancherel's theorem and the definition of $\mathscr{H}_2^\alpha(\mathbb{R}^n)$, one has

$$\begin{aligned}
\|x(\cdot) - T_\rho x(\cdot)\|^2_{L_2(\mathbb{R}^n)} &= \frac{1}{(2\pi)^n} \|Fx(\cdot) - FT_\rho x(\cdot)\|^2_{L_2(\mathbb{R}^n)} \\
&= \frac{1}{(2\pi)^n} \int_{|\sigma| \geqslant \rho} |Fx(\sigma)|^2 \, d\sigma \\
&= \frac{1}{(2\pi)^n} \int_{|\sigma| \geqslant \rho} (1+|\sigma|^2)^{-\alpha} (1+|\sigma|^2)^\alpha |Fx(\sigma)|^2 \, d\sigma \\
&\leqslant \frac{(1+\rho^2)^{-\alpha}}{(2\pi)^n} \int_{\mathbb{R}^n} |(1+|\sigma|^2)^{\alpha/2} Fx(\sigma)|^2 \, d\sigma \\
&= \frac{(1+\rho^2)^{-\alpha}}{(2\pi)^n} \|FI_\alpha x(\cdot)\|^2_{L_2(\mathbb{R}^n)} \\
&= (1+\rho^2)^{-\alpha} \|I_\alpha x(\cdot)\|^2_{L_2(\mathbb{R}^n)} \\
&= (1+\rho^2)^{-\alpha} \|x(\cdot)\|^2_{\mathscr{H}_2^\alpha(\mathbb{R}^n)} \leqslant (1+\rho^2)^{-\alpha}.
\end{aligned} \tag{3.11}$$

Evidently, $\mathrm{Im}\, T_\rho \subset \mathscr{G}_\rho(\mathbb{R}^n)$ for any $\rho > 0$. Let $\hat{\rho} = 2\sqrt{\pi}(\Gamma(n/2)+1)\nu)^{1/n}$. Then $V_n(\hat{\rho})/(2\pi)^n = \nu$. Hence, by (3.10), $\overline{\dim}(\mathrm{Im}\, T_{\hat{\rho}}, L_2(\mathbb{R}^n)) \leqslant \nu$. So it follows from this and (3.11) that

$$\begin{aligned}
\bar{d}_\nu(B\mathscr{H}_2^\alpha(\mathbb{R}^n), L_2(\mathbb{R}^n)) &\leqslant d(B\mathscr{H}_2^\alpha(\mathbb{R}^n), \mathrm{Im}\, T_{\hat{\rho}}, L_2(\mathbb{R}^n)) \\
&\leqslant \sup_{x(\cdot) \in B\mathscr{H}_2^\alpha(\mathbb{R}^n)} \|x(\cdot) - T_{\hat{\rho}} x(\cdot)\|_{L_2(\mathbb{R}^n)} \\
&\leqslant \left(1 + 4\pi\left(\Gamma\left(\frac{n}{2}+1\right)\nu\right)^{2/n}\right)^{-\alpha/2}.
\end{aligned}$$

The lower bound. Let $\rho > 0$ be such that $V_n(\rho)/(2\pi)^n > \nu$. It is obvious that there exist a positive integer N, sets $\xi_s + \Delta_\sigma \subset R^n$, $s = 1, ..., N$, where $\xi_s \in \mathbb{R}^n$, $s = 1, ..., N$, $\sigma > 0$, and $\Delta_\sigma := \{t = (t_1, ..., t_n) \in \mathbb{R}^n \mid |t_i| \leqslant \sigma, i = 1, ..., n\}$ so that $\operatorname{int}(\xi_i + \Delta_\sigma) \cap \operatorname{int}(\xi_j + \Delta_\sigma) = \varnothing$, $i \neq j$, $\bigcup_{s=1}^N (\xi_s + \Delta_\sigma) \subset \rho B\mathbb{R}^n$ and $\operatorname{mes}(\bigcup_{s=1}^N (\xi_s + \Delta_\sigma))/(2\pi)^n = N(2\sigma)^n/(2\pi)^n > \nu$.

Choose $\mu_1 \in (0, \sigma)$ such that $N(2(\sigma - \mu_1))^n/(2\pi)^n > \nu$ and let $0 < \mu < \mu_1 < \sigma$ and $k = (k_1, ..., k_n) \in \mathbb{Z}^n$. Consider the function

$$\varphi_{k,\sigma}(t_1, ..., t_n) := \prod_{j=1}^n \frac{\sin(\sigma - \mu)(t_j - k_j\pi/(\sigma - \mu)) \sin \mu(t_j - k_j\pi/(\sigma - \mu))}{\mu(\sigma - \mu)(t_j - k_j\pi/(\sigma - \mu))^2}.$$

It is easy to verify that $\varphi_{k,\sigma}(\cdot) \in \mathscr{B}_{\bar{\sigma},2}(\mathbb{R}^n)$, where $\bar{\sigma} = (\sigma, ..., \sigma)$. Then, by the Paley–Wiener theorem, $F\varphi_{k,\sigma}(t) = 0$ a.e. on $\mathbb{R}^n \backslash \Delta_\sigma$.

Let $a > 0$. Set

$$Q(a) := \operatorname{span}\{\varphi_{k,\sigma}(t)\, e^{i\xi_s t} \mid |k_j| \leqslant [a(\sigma - \mu_1)/\pi], j = 1, ..., n, s = 1, ..., N\}.$$

If $x(\cdot) \in Q(a)$ we see that $Fx(t) = 0$ a.e. on $\mathbb{R}^n \backslash (\bigcup_{s=1}^N (\xi_s + \Delta_\sigma))$. In particular, $Q(a) \subset \mathscr{G}_\rho(\mathbb{R}^n)$.

Next, there is an $a_0 > 0$ such that for each $a \geqslant a_0$ and any $x(\cdot) \in Q(a)$ the inequality

$$\|x(\cdot)\|_{L_2(\mathbb{R}^n)} \leqslant \eta(a)\, \|x(\cdot)\|_{L_2([-a,a]^n)} \tag{3.12}$$

holds true, where $\eta(a) > 0$ and $\eta(a) \to 1$ as $a \to \infty$.

For $n = 1$, (3.12) was proved in [7]. The argument in the general case is similar.

We now show that

$$\bar{d}_\nu(\mathscr{G}_\rho(\mathbb{R}^n) \cap BL_2(\mathbb{R}^n), L_2(\mathbb{R}^n)) \geqslant 1. \tag{3.13}$$

Let $L \in \operatorname{Lin}_c(L_2(\mathbb{R}^n))$, $\overline{\dim}(L, L_2(\mathbb{R}^n)) \leqslant \nu$, $a \geqslant a_0$, $S(a)$ be the restriction of $Q(a)$ to $[-a, a]^n$, and $x_a(\cdot) \in S(a) \cap (\eta(a))^{-1} BL_2([-a, a]^n)$. Because $x_a(\cdot)$ is an analytic function, there is a unique function $x(\cdot) \in Q(a)$ such that $x(\cdot)|_{[-a,a]^n} = x_a(\cdot)$. Hence, by (3.12), $\|x(\cdot)\|_{L_2(\mathbb{R}^n)} \leqslant 1$, that is, $x(\cdot) \in \mathscr{G}_\rho(\mathbb{R}^n) \cap BL_2(\mathbb{R}^n)$.

By an argument similar to the proof of Theorem 1.1, we have (see (3.3))

$$(1 + \varepsilon)\, d(\mathscr{G}_\rho(\mathbb{R}^n) \cap BL_2(\mathbb{R}^n), L, L_2(\mathbb{R}^n)) \geqslant d(x_a(\cdot), M(a, \varepsilon), L_2([-a, a]^n)) - \varepsilon, \tag{3.14}$$

where $\varepsilon > 0$, $M(a, \varepsilon)$ is a subspace of $L_2([-a, a]^n)$, and $\dim M(a, \varepsilon) \leqslant K_\varepsilon(a, L, L_2(\mathbb{R}^n))$.

By taking the supremum over such $x_a(\cdot)$, we obtain

$$\begin{aligned}(1+\varepsilon)\, & d(\mathscr{G}_\rho(\mathbb{R}^n) \cap BL_2(\mathbb{R}^n), L, L_2(\mathbb{R}^n)) \\ & \geqslant (\eta(a))^{-1}\, d(S(a) \cap BL_2([-a, a]^n), M(a, \varepsilon), L_2([-a, a]^n)) - \varepsilon. \end{aligned} \tag{3.15}$$

It is clear that $\dim S(a) = \dim Q(a) = N(2[a(\sigma - \mu_1)/\pi] + 1)^n$. By assumption $\lim_{a \to \infty} (\dim S(a)/(2a)^n) = N(2(\sigma - \mu_1))^n/(2\pi)^n =: \nu_1 > \nu$. Let $\delta > 0$ satisfy $\nu_1 - \delta > \nu + \delta$ and $\{a_s\}_{s \in \mathbb{N}}$ be a sequence for which

$$\liminf_{a \to \infty} \frac{K_\varepsilon(a, L, L_2(\mathbb{R}^n))}{(2a)^n} = \lim_{s \to \infty} \frac{K_\varepsilon(a_s, L, L_2(\mathbb{R}^n))}{(2a_s)^n}.$$

Then there exists an integer S_0 such that $K_\varepsilon(a_s, L, L_2(\mathbb{R}^n)) \leqslant (\nu + \delta)(2a_s)^n$ and $\dim S(a_s) \geqslant (\nu_1 - \delta)(2a_s)^n$ for all $s \geqslant s_0$. That is,

$$\dim M(a_s, \varepsilon) \leqslant K_\varepsilon(a_s, L, L_2(\mathbb{R}^n)) < \dim S(a_s).$$

Hence, by Tikhomirov's theorem (let X be a normed linear space, $L \in \mathrm{Lin}(X)$, and $\dim L = n+1$; then $d_n(L \cap BX, X) = 1$), we have

$$d(S(a_s) \cap BL_2([-a_s, a_s]^n), M(a_s, \varepsilon), L_2([-a_s, a_s]^n)) \geqslant 1$$

for all $s \geqslant s_0$.

From this and (3.15) and since $\eta(a) \to 1$ as $a \to \infty$, we obtain (3.13).

Next, the Bernstein-type inequality

$$\|x(\cdot)\|_{\mathscr{H}_2^\alpha(\mathbb{R}^n)} \leqslant (1+\rho^2)^{\alpha/2}\, \|x(\cdot)\|_{L_2(\mathbb{R}^n)} \tag{3.16}$$

holds true for all $x(\cdot) \in \mathscr{G}_\rho(\mathbb{R}^n)$.

Indeed, if $x(\cdot) \in \mathscr{G}_\rho(\mathbb{R}^n)$, then, by Plancherel's theorem,

$$\begin{aligned}\|x(\cdot)\|^2_{\mathscr{H}_2^\alpha(\mathbb{R}^n)} &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} ((1+|\lambda|^2)^{\alpha/2}\, |Fx(\lambda)|)^2\, d\lambda \\ &= \frac{1}{(2\pi)^n} \int_{|\lambda| \leqslant \rho} (1+|\lambda|^2)^\alpha\, |Fx(\lambda)|^2\, d\lambda \\ &\leqslant \frac{1}{(2\pi)^n} (1+\rho^2)^\alpha \int_{\mathbb{R}^n} |Fx(\lambda)|^2\, d\lambda \\ &= (1+\rho^2)^\alpha\, \|x(\cdot)\|^2_{L_2(\mathbb{R}^n)}.\end{aligned}$$

Inequality (3.16) means that $\mathscr{G}_\rho(\mathbb{R}^n) \cap (1+\rho^2)^{-\alpha/2} BL_2(\mathbb{R}^n) \subset B\mathscr{H}_2^\alpha(\mathbb{R}^n)$. It follows from this and (3.13) that

$$\bar{d}_\nu(B\mathscr{H}_2^\alpha(\mathbb{R}^n), L_2(\mathbb{R}^n)) \geqslant (1+\rho^2)^{-\alpha/2} \bar{d}_\nu(\mathscr{G}_\rho(\mathbb{R}^n) \cap BL_2(\mathbb{R}^n), L_2(\mathbb{R}^n)) \geqslant (1+\rho^2)^{-\alpha/2}.$$

Since this is true of every $\rho > 0$ such that $V_n(\rho)/(2\pi)^n > \nu$ we deduce that

$$\bar{d}_\nu(B\mathscr{H}_2^\alpha(\mathbb{R}^n), L_2(\mathbb{R}^n)) \geqslant (1+\hat{\rho}^2)^{-\alpha/2} = \left(1 + 4\pi\left(\Gamma\left(\frac{n}{2}+1\right)\nu\right)^{2/n}\right)^{-\alpha/2}.$$

Theorem 1.2 is proved.

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